

Class of integrable diffusion-reaction processes

M. Alimohammadi^{1,2,*} and N. Ahmadi¹

¹*Physics Department, University of Tehran, North Karegar Avenue, Tehran, Iran*

²*Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 5531, Tehran 19395, Iran*

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We consider a process in which there are two types of particles, A and B , on an infinite one-dimensional lattice. The particles hop to their adjacent sites, like the totally asymmetric exclusion process (ASEP), and have also the following interactions: $A + B \rightarrow B + B$ and $B + A \rightarrow B + B$, which all occur with equal rate. We study this process by imposing four boundary conditions on the ASEP master equation. It is shown that this model is integrable, in the sense that its N -particle S matrix is factorized into a product of two-particle S matrices and, more importantly, the two-particle S matrix satisfies the quantum Yang-Baxter equation. Using the coordinate Bethe-ansatz, the N -particle wave functions and the two-particle conditional probabilities are found exactly. Further, by imposing four reasonable physical conditions on two-species diffusion-reaction processes (where the most important ones are the equality of the reaction rates and the conservation of the number of particles in each reaction), we show that among the 4096 types of interactions which have these properties and can be modeled by a master equation and an appropriate set of boundary conditions, there are only 28 independent interactions which are integrable. We find all these interactions and also their corresponding wave functions. Some of these may be new solutions of the quantum Yang-Baxter equation.

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I. INTRODUCTION

For nonequilibrium systems in low dimensions, an understanding can often be gained by studying rather simple models [1–4]. One of the important examples of these systems is reaction-diffusion processes on a one-dimensional lattice, the dynamics of which are fully specified by their master equation [5,6]. In some cases, it is possible to solve the master equation exactly. In recent years, there has been enormous progress in the field of exactly solvable nonequilibrium processes. These developments were mainly triggered by the observation that the Liouville operator of certain $(1+1)$ -dimensional reaction-diffusion models may be related to Hamiltonians of previously known quantum spin systems [7,8].

One of the simplest examples of reaction-diffusion processes is the asymmetric simple exclusion processes (ASEP) [2,9,10], which has been used to describe various problems in different fields of interest, such as the kinetics of bipolymerization [11], dynamical models of interface growth [12], and traffic models [13]. The totally ASEP model has been solved exactly by imposing the appropriate boundary condition on the probabilities that appear in the master equation [14]. The totally ASEP model describes a process in which each lattice site can be occupied by at most one particle and the particle hops with rate one to its right-neighboring site if it is not already occupied, otherwise the attempted move is rejected.

There are some other interesting and more complicated processes which can be solved by the method developed in [14], namely by choosing a suitable boundary condition for the master equation. For example, it has been shown that the so-called “generalized totally ASEP model” can be solved

exactly by this method [15]. In this model, even if the right-neighboring site of a particle is occupied, the particle hops to the next right site by pushing all the neighboring particles to their next right sites, with a rate which depends on the number of right-neighboring particles. This model has been further generalized in [16] by considering both the right and left hopping of the particles.

In this paper we are going to consider a class of integrable models in which there are *two* species of particles which can hop to their right-neighboring sites if those are not occupied, and can also interact with each other if they are in adjacent sites. The details of this nearest-neighboring interaction depend on the specific considered model (see [17–19] for some recent works on two- and three-species reaction-diffusion processes). The important point in integrable reaction-diffusion processes with more than one type of particle is that, as we will show, the two-particle S matrix of the reaction, which specifies the N -point functions, must satisfy the quantum Yang-Baxter equation (QYBE). Therefore, as we expect, the number of integrable models, in the sense that their N -particle S matrix can be factorized into a product of two-particle S matrices, is very few. In this paper we will find all two-species integrable reaction-diffusion processes which have the following properties: (i) the particles hop to their right-neighboring sites if these sites are not occupied; (ii) the interaction occurs only between nearest-neighbor particles; (iii) the particles can be annihilated or created, with the only restriction being that the total number of particles is fixed; and (iv) all the interactions, including diffusions, occur with the same rate. We show that among the $2^{12} = 4096$ types of interactions which have the above-mentioned properties and can be modeled by a master equation and an appropriate boundary condition, there are only 42 interactions which are integrable (their two-particle S matrices satisfy the QYBE), and from these, only 28 of them are independent. Some of these may be new solutions of QYBE.

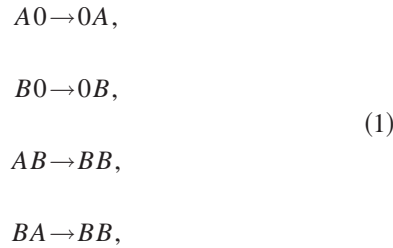
*Email address: alimohmd@theory.ipm.ac.ir

The plan of the paper is as follows. In Sec. II, we introduce the first kind of interaction, which was our initial motivation in this work, in which besides the usual hopping, the two types of particles interact as follows: $A + B \rightarrow B + B$ and $B + A \rightarrow B + B$. We show that this interaction can be modeled by the usual master equation of ASEP and four boundary conditions. We also show that the model is integrable. Note that one can look at this model [see Eq. (1)] as a simple one-dimensional model of the spread of disease. If we consider the A particles to be the healthy individuals and the B particles the diseased ones, then we expect that when the A and B particles are near each other, the healthy one may become diseased (in other words, B transmits disease to A). In Sec. III, we compute the exact two-particle conditional probabilities of this interaction and study the long-time behavior of these probabilities. And finally in Sec. IV, we investigate the class of integrable models which have the four above-mentioned properties, and we deduce that there are 28 different models, of which the totally ASEP model and our first model introduced in Sec. II are two of them.

II. $AB \rightarrow BB$ AND $BA \rightarrow BB$ REACTION-DIFFUSION PROCESS

A. The master equation

Suppose there are N particles of two types A and B on an infinite one-dimensional lattice, with interactions



all occurring with *equal* rate, which can be scaled to 1. In Eq. (1), we denote the vacancy by notation 0. The basic quantities we are interested in are the probabilities $P_{\alpha_1, \alpha_2, \dots, \alpha_N}(x_1, x_2, \dots, x_N; t)$ for finding at time t a particle of type α_1 at site x_1 , a particle of type α_2 at site x_2 , etc. Each α_i can be A or B . Following [14], we take these functions to define probabilities only in the physical region $x_1 < x_2 < \dots < x_N$. The surfaces where any of the two adjacent coordinates are equal are the boundaries of the physical region. In the subset of the physical region where $x_{i+1} - x_i > 1$, $\forall i$, we have only hopping of the particles and therefore the master equation can be written as

$$\begin{aligned} \frac{\partial}{\partial t} P_{\alpha_1, \alpha_2, \dots, \alpha_N}(x_1, x_2, \dots, x_N; t) \\ = P_{\alpha_1, \dots, \alpha_N}(x_1 - 1, x_2, \dots, x_N; t) \\ + \dots + P_{\alpha_1, \dots, \alpha_N}(x_1, x_2, \dots, x_N - 1; t) \\ - NP_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N; t). \end{aligned} \quad (2)$$

As is clear from Eq. (2), when $x_{i+1} = x_i + 1$ for some i 's, one or more of the probability functions go out from the physical

region and therefore Eq. (2) has to be supplemented by some boundary conditions. The particular choice of the boundary condition depends on the details of the interactions of particles. For example, it can be shown that in the totally ASEP model, the suitable boundary condition is [14]

$$P(x, x) = P(x, x + 1), \quad \forall x, \quad (3)$$

in which the time variable and also all the other coordinates have been suppressed for simplicity. The master equation (2) and the boundary condition (3) replace the very large number of equations which one should write by considering the multitude of cases which arises in different possible configurations.

To model the interaction (1), we claim that the suitable boundary conditions are

$$P_{AA}(x, x) = P_{AA}(x, x + 1), \quad (4)$$

$$P_{BB}(x, x) = P_{BB}(x, x + 1) + P_{AB}(x, x + 1) + P_{BA}(x, x + 1), \quad (5)$$

$$P_{AB}(x, x) = P_{BA}(x, x) = 0, \quad (6)$$

where we have again suppressed the positions of all the other particles. By looking at Eq. (1), it is obvious that if we have only A particles, the process is exactly the same as totally ASEP. This is the explanation for Eq. (4), which is the same as Eq. (3). To justify the other three boundary conditions, we provide a few examples in the two- and three-particle sectors, instead of giving a general proof.

First we consider the two-particle sector, for example $P_{BA}(x, x + 1)$. From master equation (2) we have

$$\frac{\partial}{\partial t} P_{BA}(x, x + 1) = P_{BA}(x - 1, x) + P_{BA}(x, x) - 2P_{BA}(x, x + 1). \quad (7)$$

Using Eq. (6), Eq. (7) reduces to

$$\frac{\partial}{\partial t} P_{BA}(x, x + 1) = P_{BA}(x - 1, x) - 2P_{BA}(x, x + 1). \quad (8)$$

This is exactly what we expect, as the source of configuration $(0BA0)$ is $(B0A0)$ and its sinks are two configurations $(0B0A)$ and $(0BB0)$. The second example is $P_{BB}(x, x + 1)$. Using again the master equation (2) and the boundary condition (5), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} P_{BB}(x, x + 1) = P_{BB}(x - 1, x) + P_{AB}(x, x + 1) \\ + P_{BA}(x, x + 1) - P_{BB}(x, x + 1). \end{aligned} \quad (9)$$

This equation also predicts the true source and sink terms, because $(0BB0)$ has three sources $(B0B0)$, $(0AB0)$, and $(0BA0)$ and only one sink $(0B0B)$. As a three-particle sector example, let us consider the most nontrivial case $P_{BBB}(x, x + 1, x + 2)$. Using Eqs. (3) and (5), we find

$$\begin{aligned}
\frac{\partial}{\partial t} P_{BBB}(x, x+1, x+2) &= P_{BBB}(x-1, x+1, x+2) \\
&+ P_{BBB}(x, x, x+2) + P_{BBB}(x, x+1, x+1) - 3P_{BBB}(x, x+1, x+2) \\
&= P_{BBB}(x-1, x+1, x+2) + 2P_{BAB}(x, x+1, x+2) + P_{BBA}(x, x+1, x+2) \\
&+ P_{ABB}(x, x+1, x+2) - P_{BBB}(x, x+1, x+2). \tag{10}
\end{aligned}$$

This is also the true equation, because the configuration (0BBB0) has five sources, namely (B0BB0), 2(0BAB0), (0BBA0), and (0ABBO) and one sink (0BB0B). The reason for the factor 2 in (0BAB0) is that (0BA...) can go to (0BB...) and also (...AB0) can go to (...BB0).

B. The Bethe ansatz solution

We now try to solve the master equation (2) with boundary conditions (4)–(6) by the Bethe ansatz method. If we define $\Psi_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N)$ through

$$P_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N; t) = e^{-\epsilon_N t} \Psi_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N) \tag{11}$$

and substitute Eq. (11) in the master equation (2) and boundary conditions (4)–(6), we find

$$\Psi_{\alpha_1, \dots, \alpha_N}(x_1-1, x_2, \dots, x_N) + \dots + \Psi_{\alpha_1, \dots, \alpha_N}(x_1, x_2, \dots, x_N-1) = (N - \epsilon_N) \Psi_{\alpha_1, \dots, \alpha_N}(x_1, x_2, \dots, x_N) \tag{12}$$

and

$$\begin{aligned}
\Psi_{AA}(x, x) &= \Psi_{AA}(x, x+1), \\
\Psi_{BB}(x, x) &= \Psi_{BB}(x, x+1) + \Psi_{AB}(x, x+1) + \Psi_{BA}(x, x+1), \\
\Psi_{AB}(x, x) &= \Psi_{BA}(x, x) = 0.
\end{aligned} \tag{13}$$

Following [20], it becomes easier if we use a compact notation as follows: Ψ is a tensor of rank N with components $\Psi_{\alpha_1, \dots, \alpha_N}(x_1, \dots, x_N)$. Then the boundary conditions (13) can be written as

$$\Psi(\dots, \xi, \xi, \dots) = \mathbf{b}_{k, k+1} \Psi(\dots, \xi, \xi+1, \dots), \tag{14}$$

where $\mathbf{b}_{k, k+1}$ is the embedding of \mathbf{b} [the matrix derived from Eq. (13)] in the location k and $k+1$:

$$\mathbf{b}_{k, k+1} = \mathbf{1} \otimes \dots \otimes \underbrace{\mathbf{b}}_{k, k+1} \otimes \dots \otimes \mathbf{1}, \tag{15}$$

with

$$\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \tag{16}$$

To solve Eq. (12), we write the coordinate Bethe ansatz for Ψ in the form

$$\Psi(x_1, \dots, x_N) = \sum_{\sigma} \mathbf{A}_{\sigma} e^{i\sigma(\mathbf{p}) \cdot \mathbf{x}}, \tag{17}$$

where \mathbf{x} and \mathbf{p} denote the N -tuple coordinates and momenta, respectively, the summation runs over all the elements of the permutation group, and the \mathbf{A}_{σ} 's (tensors of rank N) are coefficients that must be determined from boundary condition (14). Inserting Eq. (17) into (12) yields

$$\sum_{\sigma} \mathbf{A}_{\sigma} e^{i\sigma(\mathbf{p}) \cdot \mathbf{x}} (e^{-i\sigma(p_1)} + \dots + e^{-i\sigma(p_N)} + \epsilon_N - N) = 0, \tag{18}$$

from which one can find the eigenvalues ϵ_N as

$$\epsilon_N = \sum_{k=1}^N (1 - e^{-ip_k}). \tag{19}$$

To find the coefficients \mathbf{A}_{σ} , we insert the wave function (17) in Eq. (14), which yields

$$\begin{aligned}
\sum_{\sigma} e^{i \sum_{j \neq k, k+1} \sigma(p_j) x_j + i[\sigma(p_k) + \sigma(p_{k+1})] \xi} \\
\times [(1 - e^{i\sigma(p_{k+1})}) \mathbf{b}_{k, k+1}) \mathbf{A}_{\sigma}] = 0. \tag{20}
\end{aligned}$$

We note that the first part of the above equation is symmetric with respect to the interchange of p_k and p_{k+1} , so if we symmetrize the bracket with respect to this interchange, we obtain

$$(1 - e^{i\sigma(p_{k+1})}) \mathbf{b}_{k, k+1}) \mathbf{A}_{\sigma} + (1 - e^{i\sigma(p_k)}) \mathbf{b}_{k, k+1}) \mathbf{A}_{\sigma \sigma_k} = 0, \tag{21}$$

where σ_k is an element of permutation group which only interchanges p_k and p_{k+1} , and $\sigma \sigma_k$ stands for the product of two group elements σ and σ_k . Thus we obtain

$$\mathbf{A}_{\sigma \sigma_k} = \mathbf{S}_{k, k+1}(\sigma(p_k), \sigma(p_{k+1})) \mathbf{A}_{\sigma}, \tag{22}$$

where

$$\mathbf{S}_{k,k+1}(z_1, z_2) = \mathbf{1} \otimes \cdots \otimes \underbrace{S(z_1, z_2)}_{k, k+1} \otimes \cdots \otimes \mathbf{1}, \quad (23)$$

and

$$S(z_1, z_2) = -(\mathbf{1} - z_1 \mathbf{b})^{-1} (\mathbf{1} - z_2 \mathbf{b}). \quad (24)$$

In the above equations, we have denoted e^{ip_k} by z_k . In this way, we can calculate all the coefficients \mathbf{A}_σ 's in term of \mathbf{A}_1 by using Eq. (22), and *it seems that* the problem is solved for the arbitrary boundary condition \mathbf{b} matrix. But this is not the case and we should moreover check the consistency of the solutions, which is highly nontrivial and depends on the details of the interaction, i.e., the elements of the \mathbf{b} matrix [20].

To see this, let us find two coefficients $\mathbf{A}_{\sigma_1 \sigma_2 \sigma_1}$ and $\mathbf{A}_{\sigma_2 \sigma_1 \sigma_2}$. Note that $\sigma_1 \sigma_2 \sigma_1$ and $\sigma_2 \sigma_1 \sigma_2$ are equal as the elements of the permutation group, that is, both of them when acting on $(p_1, p_2, p_3, p_4, \dots)$ will result in

$$(p_1, p_2, p_3, p_4, \dots) \rightarrow (p_3, p_2, p_1, p_4, \dots). \quad (25)$$

So we should have

$$\mathbf{A}_{\sigma_1 \sigma_2 \sigma_1} = \mathbf{A}_{\sigma_2 \sigma_1 \sigma_2}. \quad (26)$$

Now using Eq. (22), we have

$$\begin{aligned} \mathbf{A}_{\sigma_1 \sigma_2 \sigma_3} &= S_{12}(\sigma_1 \sigma_2(p_1), \sigma_1 \sigma_2(p_2)) \mathbf{A}_{\sigma_1 \sigma_2} \\ &= S_{12}(p_2, p_3) \mathbf{A}_{\sigma_1 \sigma_2} \\ &= S_{12}(p_2, p_3) S_{23}(\sigma_1(p_2), \sigma_1(p_3)) \mathbf{A}_{\sigma_1} \\ &= S_{12}(p_2, p_3) S_{23}(p_1, p_3) \mathbf{A}_{\sigma_1} \\ &= S_{12}(p_2, p_3) S_{23}(p_1, p_3) S_{12}(p_1, p_2) \mathbf{A}_1, \end{aligned} \quad (27)$$

and in the same way,

$$\mathbf{A}_{\sigma_2 \sigma_1 \sigma_2} = S_{23}(p_1, p_2) S_{12}(p_1, p_3) S_{23}(p_2, p_3) \mathbf{A}_1. \quad (28)$$

Therefore, Eq. (26) yields

$$\begin{aligned} &S_{12}(p_2, p_3) S_{23}(p_1, p_3) S_{12}(p_1, p_2) \\ &= S_{23}(p_1, p_2) S_{12}(p_1, p_3) S_{23}(p_2, p_3), \end{aligned} \quad (29)$$

which is the familiar quantum Yang-Baxter equation. Therefore, we must check if the \mathbf{S} matrices defined in Eqs. (23) and (24), with \mathbf{b} from Eq. (16), satisfy Eq. (29) or not.

Using Eq. (16), it can be shown that Eq. (24) is equal to

$$S(z, w) = \frac{1}{z-1} \begin{bmatrix} 1-w & 0 & 0 & 0 \\ 0 & 1-z & 0 & 0 \\ 0 & 0 & 1-z & 0 \\ 0 & z-w & z-w & 1-w \end{bmatrix}, \quad (30)$$

and if we define $z = e^{ip_1}$, $w = e^{ip_2}$, and $t = e^{ip_3}$, Eq. (29) can be written as

$$\begin{aligned} &[S(w, t) \otimes \mathbf{1}][\mathbf{1} \otimes S(z, t)][S(z, w) \otimes \mathbf{1}] \\ &= [\mathbf{1} \otimes S(z, w)][S(z, t) \otimes \mathbf{1}][\mathbf{1} \otimes S(w, t)]. \end{aligned} \quad (31)$$

Now it is not too difficult to show that the matrix (30) really satisfies Eq. (31), and therefore the solutions of interactions (1) are wave functions (17) with the coefficients that can be found by Eq. (22).

III. TWO-PARTICLE CONDITIONAL PROBABILITIES

In this section we want to calculate the two-particle conditional probabilities $P(\alpha_1, \alpha_2, x_1, x_2; t | \beta_1, \beta_2, y_1, y_2; 0)$, which is the probability of finding α_1 at site x_1 and particle α_2 at site x_2 at time t , if at time $t=0$ we have the particle β_1 at site y_1 and particle β_2 at site y_2 . As has been discussed in [14] and [15], we should take a linear combination of eigenfunctions $P(x_1, x_2)$ [from Eqs. (11) and (17)] with suitable coefficients to find these two-particle conditional probabilities. Therefore,

$$\begin{aligned} &\begin{pmatrix} P_{AA} \\ P_{AB} \\ P_{BA} \\ P_{BB} \end{pmatrix}(\mathbf{x}; t | \beta, \mathbf{y}; 0) \\ &= \int f(p_1, p_2) e^{-\epsilon_2 t} \Psi(x_1, x_2) dp_1 dp_2 \\ &= \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} e^{-i\mathbf{p} \cdot \mathbf{y}} \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{i(p_1 x_1 + p_2 x_2)} \right. \\ &\quad \left. + S_{12}(p_1, p_2) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{i(p_2 x_1 + p_1 x_2)} \right\} dp_1 dp_2. \end{aligned} \quad (32)$$

In the above expansion, $P_{\alpha_1 \alpha_2}(\mathbf{x}; t | \beta, \mathbf{y}; 0)$ is $P(\alpha_1, \alpha_2, x_1, x_2; t | \beta_1, \beta_2, y_1, y_2; 0)$, $f(p_1, p_2)$ is the coefficient of expansion which in the second equality we choose $[1/(2\pi)^2] e^{-i\mathbf{p} \cdot \mathbf{y}}$, and $\epsilon_2 = 2 - e^{-ip_1} - e^{-ip_2}$ [see Eq. (19)]. $\Psi(x_1, x_2)$ is the two-particle wave function where from Eqs. (17) and (22) we obtain

$$\begin{aligned} \Psi(x_1, x_2) &= \mathbf{A}_1 e^{i(p_1 x_1 + p_2 x_2)} + \mathbf{A}_{\sigma_1} e^{i\sigma_1(\mathbf{p}) \cdot \mathbf{x}} \\ &= \mathbf{A}_1 e^{i(p_1 x_1 + p_2 x_2)} + S_{12}(p_1, p_2) \mathbf{A}_1 e^{i(p_1 x_2 + p_2 x_1)}. \end{aligned} \quad (33)$$

In the two-particle sector, \mathbf{A}_1 is a four-column vector whose components must be determined by initial conditions, and $S_{12}(p_1, p_2)$ can be read from Eq. (30):

$$S_{12}(p_1, p_2) = \begin{pmatrix} s' & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & s & s & s' \end{pmatrix}, \tag{34}$$

with

$$s' = \frac{1 - e^{ip_2}}{e^{ip_1} - 1}, \tag{35}$$

$$s = \frac{e^{ip_1} - e^{ip_2}}{e^{ip_1} - 1}.$$

Let us first calculate Eq. (32) irrespective of the initial conditions, that is, for arbitrary a, b, c, d . Substituting Eq. (34) in Eq. (32), we find

$$\begin{pmatrix} P_{AA} \\ P_{AB} \\ P_{BA} \\ P_{BB} \end{pmatrix}(\mathbf{x}; t | \beta, \mathbf{y}; 0) = \begin{pmatrix} a[F_1(t) + F_4(t)] \\ b[F_1(t) - F_2(t)] \\ c[F_1(t) - F_2(t)] \\ d[F_1(t) + F_4(t)] + (b + c)F_3(t) \end{pmatrix}, \tag{36}$$

in which

$$F_1(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} dp_1 dp_2, \tag{37}$$

$$F_2(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} e^{i(\tilde{\mathbf{p}} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{y})} dp_1 dp_2, \tag{38}$$

$$F_3(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} \frac{e^{ip_1} - e^{ip_2}}{e^{ip_1} - 1} e^{i(\tilde{\mathbf{p}} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{y})} dp_1 dp_2, \tag{39}$$

$$F_4(t) = \frac{1}{(2\pi)^2} \int e^{-\epsilon_2 t} \frac{1 - e^{ip_2}}{e^{ip_1} - 1} e^{i(\tilde{\mathbf{p}} \cdot \mathbf{x} - \mathbf{p} \cdot \mathbf{y})} dp_1 dp_2, \tag{40}$$

where in the above equations we have suppressed the \mathbf{x} and \mathbf{p} dependence of F_i 's for simplicity and $\tilde{\mathbf{p}} = (p_2, p_1)$. To avoid the singularity in s and s' , we set $p_1 \rightarrow p_1 + i\epsilon$, and then by some simple contour integration we find

$$F_1(0) = \delta_{x_1, y_1} \delta_{x_2, y_2}, \tag{41}$$

$$F_2(0) = F_3(0) = F_4(0) = 0,$$

and at $t \neq 0$,

$$F_1(t) = e^{-2t} \frac{t^{x_1 - y_1}}{(x_1 - y_1)!} \frac{t^{x_2 - y_2}}{(x_2 - y_2)!},$$

$$F_2(t) = e^{-2t} \frac{t^{x_2 - y_1}}{(x_2 - y_1)!} \frac{t^{x_1 - y_2}}{(x_1 - y_2)!},$$

$$F_3(t) = e^{-2t} \left\{ \frac{t^{x_1 - y_2 + 1}}{(x_1 - y_2 + 1)!} \sum_{k=0}^{\infty} \frac{t^{x_2 - y_1 + k}}{(x_2 - y_1 + k)!} - \frac{t^{x_1 - y_2}}{(x_1 - y_2)!} \sum_{k=0}^{\infty} \frac{t^{x_2 - y_1 + k + 1}}{(x_2 - y_1 + k + 1)!} \right\},$$

$$F_4(t) = e^{-2t} \left\{ \frac{t^{x_1 - y_2 + 1}}{(x_1 - y_2 + 1)!} - \frac{t^{x_1 - y_2}}{(x_1 - y_2)!} \sum_{k=0}^{\infty} \frac{t^{x_2 - y_1 + k}}{(x_2 - y_1 + k)!} \right\}. \tag{42}$$

Now we can study the different initial conditions.

A. Case of $\beta_1 = \beta_2 = A$

If at $t=0$ both particles are A type, then our initial condition is

$$\begin{pmatrix} P_{AA} \\ P_{AB} \\ P_{BA} \\ P_{BB} \end{pmatrix}(\mathbf{x}; 0 | A, A, \mathbf{y}; 0) = \begin{pmatrix} \delta_{x_1, y_1} \delta_{x_2, y_2} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{43}$$

Using Eqs. (36) and (41), we find

$$a = 1, \quad b = c = d = 0, \tag{44}$$

and therefore

$$P_{AA}(\mathbf{x}; t | A, A, \mathbf{y}; 0) = F_1(t) + F_4(t), \tag{45}$$

and all other P 's are zero. Note that Eq. (45) is exactly the same conditional probability that has been found in [14] for the simple ASEP model, as we expected.

B. Case of $\beta_1 = A, \beta_2 = B$

In this case the only nonzero element of conditional probability, at $t=0$, is $P_{AB} = \delta_{x_1, y_1} \delta_{x_2, y_2}$. Therefore, one finds

$$b = 1, \quad a = c = d = 0, \tag{46}$$

and therefore

$$P_{AB}(\mathbf{x}; t | A, B, \mathbf{y}; 0) = F_1(t) - F_2(t), \tag{47}$$

$$P_{BB}(\mathbf{x}; t | A, B, \mathbf{y}; 0) = F_3(t),$$

and $P_{AA} = P_{BA} = 0$, which is consistent with our processes (1). It can also be checked that the conservation of probability holds,

$$\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} (P_{AB} + P_{BB})(\mathbf{x}; t | A, B, \mathbf{y}; 0) = 1, \tag{48}$$

for arbitrary y_1, y_2 , and t .

C. Case of $\beta_1 = B, \beta_2 = A$

In this case, the final result is

$$P_{BA}(\mathbf{x};t|B,A,\mathbf{y};0) = F_1(t) - F_2(t),$$

$$P_{BB}(\mathbf{x};t|B,A,\mathbf{y};0) = F_3(t),$$
(49)

and $P_{AA} = P_{AB} = 0$.

D. Case of $\beta_1 = \beta_2 = B$

In this case the final result is the same as the case Sec. III A as we expected,

$$P_{BB}(\mathbf{x};t|B,B,\mathbf{y};0) = F_1(t) + F_4(t),$$
(50)

and $P_{AA} = P_{AB} = P_{BA} = 0$.

Another interesting quantity that can be calculated is the long-time behavior of these functions. The only nontrivial case is the Sec. III B (or equivalently Sec. III C). We expect that if at $t=0$ we have A and B particles (one healthy and one diseased individual), the healthy one becomes diseased finally, or we have two B particles finally. In other words, we expect that (in Sec. III B)

$$\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{AB}(\mathbf{x};t \rightarrow \infty | A, B, \mathbf{y}; 0) \rightarrow 0,$$
(51)

and

$$\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{BB}(\mathbf{x};t \rightarrow \infty | A, B, \mathbf{y}; 0) \rightarrow 1.$$
(52)

After some calculations, one can show that

$$\begin{aligned} & \sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{AB}(\mathbf{x};t|A,B,\mathbf{y};0) \\ &= e^{-2t} \left[2 \sum_{m=1}^{y_2-y_1-1} I_m(2t) + I_0(2t) + I_{y_2-y_1}(2t) \right], \end{aligned}$$
(53)

where $I_n(x)$ is the n th order Bessel function of the first kind:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{n+2k}}{k!(n+k)!}.$$
(54)

Now at $x \rightarrow \infty$, we have

$$I_n(x) \rightarrow \frac{e^x}{\sqrt{2\pi x}},$$
(55)

therefore

$$\sum_{x_2=y_2}^{\infty} \sum_{x_1=y_1}^{x_2-1} P_{AB}(\mathbf{x};t \rightarrow \infty | A, B, \mathbf{y}; 0) \rightarrow \frac{M}{\sqrt{4\pi t}},$$
(56)

which goes to zero. M is the number of the $I_n(2t)$'s in the left-hand side of Eq. (53). Using Eq. (48), we see that both limits in Eqs. (51) and (52) are satisfied.

IV. CLASS OF MODELS

Now we want to find the class of integrable models which have the four properties introduced in the Introduction. If we look at the preceding sections, we notice that all the information about the model is abbreviated in the \mathbf{b} matrix (16), because this matrix comes from the boundary conditions (4)–(6), and the latter induce our interactions. We also note that the sum of each column of \mathbf{b} matrix is 1. Now we claim that each \mathbf{b} matrix, which has the properties in which (i) the nondiagonal elements are 1 or zero and (ii) the sum of elements in each column is 1, represents a model in which its interaction(s) can be induced by the master equation (2) and the boundary condition(s) which can be read from \mathbf{b} .

The reason for the first requirement is that the nondiagonal elements are the sources of our reactions, as can be seen from the example solved in Sec. II A, and if we want all the reactions to occur with equal rate 1, the prefactors of all source terms must be one (or zero if we do not want the corresponding source terms). Note that we should take all the rates equal to each other, otherwise for the reactions we are interested in (i.e., those in which particles change their type), the factorization (11) will not yield the time-independent boundary condition(s), which is wrong.

The reason for the second requirement lies in the conservation of probability. Suppose that in the first column of \mathbf{b} , for example, the sum of the nondiagonal elements is m and the diagonal element is n . So we have m possible interactions, each of which can be a sink for AA . Therefore, if our configuration is $(0AA0)$, we must have $m+1$ sinks, one sink for diffusion $(0AA0) \rightarrow (0A0A)$, and m sinks for reactions $(0AA0) \rightarrow (0\alpha\beta 0)$. Now we consider the master equation for this process:

$$\begin{aligned} \frac{\partial}{\partial t} P_{AA}(x, x+1) &= P_{AA}(x-1, x+1) + P_{AA}(x, x) \\ &\quad - 2P_{AA}(x, x+1) \\ &= P_{AA}(x-1, x+1) + \sum_1^{m'} P_{\alpha\beta}(x, x+1) \\ &\quad - (2-n)P_{AA}(x, x+1), \end{aligned}$$
(57)

in which we have supposed that the m' elements of the first row of \mathbf{b} (besides b_{11}) are different from zero and therefore the corresponding probabilities appear in $P_{AA}(x, x)$. Now as the number of sinks must be $m+1$, so also must $2-n=m+1$, which yields $n+m=1$. Therefore, the sum of the elements of the first column of \mathbf{b} must be 1. By the same reasoning, this is true for other columns.

In this way we have $2^{12}=4096$ possibilities for matrices \mathbf{b} (there are 12 nondiagonal elements, where each can be 1 or zero), where each plus master equation (2) can model a reaction-diffusion process. But as we have seen in reaction (2), these \mathbf{b} 's must be consistent with the QYBE (31). Therefore, the domain of \mathbf{b} 's is much smaller. So it is sufficient to check which of these \mathbf{b} 's [or more carefully, the S matrices that are constructed by these \mathbf{b} 's from Eq. (24)] satisfies Eq. (31). Using a symbolic manipulator (e.g., MAPLE), we found that there are 42 different \mathbf{b} 's that satisfy Eq. (31). By a closer inspection of these matrices, it can be observed that 14

of them can be obtained from the others by interchanging $A \leftrightarrow B$, so they do not represent any new physical interactions. The $42 - 14 = 28$ \mathbf{b} 's (interactions) are as follows:

$$\mathbf{b}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ pure diffusion,}$$

$$\mathbf{b}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ } AB \rightarrow BA,$$

$$\mathbf{b}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \text{ } BA \rightarrow BB,$$

$$\mathbf{b}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \text{ } AB \rightarrow BB,$$

$$\mathbf{b}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \left. \begin{matrix} AB \\ BA \end{matrix} \right\} \rightarrow BB,$$

$$\mathbf{b}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \text{ } AB \rightarrow BA, \text{ } AA \rightarrow BB,$$

$$\mathbf{b}_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \text{ } AA \rightarrow BA, \text{ } AB \rightarrow BB,$$

$$\mathbf{b}_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \left. \begin{matrix} AA \\ AB \end{matrix} \right\} \rightarrow BA,$$

$$\mathbf{b}_9 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \text{ } BA \rightarrow AB, \text{ } AA \rightarrow BB,$$

$$\mathbf{b}_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ } AA \rightarrow AB, \text{ } BB \rightarrow BA,$$

$$\mathbf{b}_{11} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \text{ } BA \rightarrow AA, \text{ } AB \rightarrow BB,$$

$$\mathbf{b}_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \left. \begin{matrix} AA \\ BA \end{matrix} \right\} \rightarrow AB,$$

$$\mathbf{b}_{13} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \text{ } AB \rightarrow AA, \text{ } BA \rightarrow BB,$$

$$\mathbf{b}_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \text{ } AA \rightarrow AB, \text{ } BA \rightarrow BB,$$

$$\mathbf{b}_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ } BB \rightarrow AB, \text{ } AA \rightarrow BA,$$

$$\mathbf{b}_{16} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \left. \begin{matrix} AB \rightarrow BB, \\ BB \\ AB \end{matrix} \right\} \rightarrow BA,$$

$$\mathbf{b}_{17} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \left. \begin{matrix} AA \\ AB \\ BA \end{matrix} \right\} \rightarrow BB,$$

$$\mathbf{b}_{18} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \left. \begin{matrix} BA \rightarrow AA, \\ AB \\ BB \end{matrix} \right\} \rightarrow BA,$$

$$\mathbf{b}_{19} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \left. \begin{matrix} AB \\ BA \end{matrix} \right\} \rightarrow AA, \text{ } BB \rightarrow \left\{ \begin{matrix} AB \\ BA \end{matrix} \right.$$

$$\mathbf{b}_{20} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad BA \rightarrow AA, \quad BB \rightarrow AB,$$

$$AA \rightarrow BA, \quad AB \rightarrow BB,$$

$$\mathbf{b}_{21} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad AB \rightarrow AA, \quad AA \rightarrow AB,$$

$$BB \rightarrow BA, \quad BA \rightarrow BB,$$

$$\mathbf{b}_{22} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$BB \rightarrow AA, \quad BA \rightarrow AB, \quad AB \rightarrow BA, \quad AA \rightarrow BB,$$

$$\mathbf{b}_{23} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad BA \rightarrow AA, \quad AB \rightarrow BB,$$

$$\left. \begin{matrix} BA \\ AA \end{matrix} \right\} \rightarrow AB, \quad \left. \begin{matrix} BB \\ AB \end{matrix} \right\} \rightarrow BA,$$

$$\mathbf{b}_{24} = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \quad \left. \begin{matrix} BB \\ AB \end{matrix} \right\} \rightarrow AA, \quad \left\{ \begin{matrix} AA \\ BA \end{matrix} \right\} \rightarrow BB,$$

$$BB \rightarrow AB, \quad AA \rightarrow BA,$$

$$\mathbf{b}_{25} = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \quad \left. \begin{matrix} BB \\ BA \end{matrix} \right\} \rightarrow AA,$$

$$\left. \begin{matrix} BB \\ BA \end{matrix} \right\} \rightarrow AB,$$

$$\left. \begin{matrix} AA \\ AB \end{matrix} \right\} \rightarrow BA, \quad \left. \begin{matrix} AA \\ AB \end{matrix} \right\} \rightarrow BB,$$

$$\mathbf{b}_{26} = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \quad \left. \begin{matrix} BB \\ AB \end{matrix} \right\} \rightarrow AA,$$

$$\left. \begin{matrix} BB \\ AB \end{matrix} \right\} \rightarrow BA, \quad \left. \begin{matrix} AA \\ BA \end{matrix} \right\} \rightarrow AB, \quad \left. \begin{matrix} AA \\ BA \end{matrix} \right\} \rightarrow BB,$$

$$\mathbf{b}_{27} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \quad \left. \begin{matrix} BA \\ AB \end{matrix} \right\} \rightarrow AA,$$

$$\left. \begin{matrix} AA \\ BB \end{matrix} \right\} \rightarrow BA, \quad \left. \begin{matrix} BB \\ AA \end{matrix} \right\} \rightarrow AB, \quad \left. \begin{matrix} AB \\ BA \end{matrix} \right\} \rightarrow BB,$$

$$\mathbf{b}_{28} = \begin{pmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix}, \quad \left. \begin{matrix} AB \\ BB \\ BA \end{matrix} \right\} \rightarrow AA,$$

$$\left. \begin{matrix} BB \\ AA \\ AB \end{matrix} \right\} \rightarrow BA, \quad \left. \begin{matrix} AA \\ BB \\ BA \end{matrix} \right\} \rightarrow AB, \quad \left. \begin{matrix} BA \\ AA \\ AB \end{matrix} \right\} \rightarrow BB.$$

It should be mentioned that in the above, the reaction processes of each \mathbf{b} have been given only and the diffusion processes (which exist in all cases) have been suppressed. Also note that \mathbf{b}_1 is the pure diffusion process of [14], and \mathbf{b}_5 is merely Eq. (16).

In all the above cases, the probabilities Ψ and \mathbf{A}_σ are given by Eqs. (11), (17), and (22), respectively. Obviously, $S(z, w)$ must be calculated from Eq. (24) for each case, and then the calculations of Sec. III can be repeated for them.

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